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PROPAGATION OF THIN PLASTIC ZONES IN THE VICINITY OF A NORMALLY SEPARATING CRACK

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A problem of the development of a plastic zone in the vicinity of a physical cut in the plain strain and stress states is posed and solved on the basis of a discrete deformation model under the assumption of an ideal elastoplastic medium. The Tresca yield condition and the ultimate plasticity condition are used in studying the plane stress state. The dependence of the plastic zone length on the external load is compared with a similar dependence obtained on the basis of the Leonov–Panasyuk–Dugdale model. In contrast to the Leonov–Panasyuk–Dugdale model, the distributions of stresses and lengths of plastic zones in the plane strain and stress states are found to be substantially different if elastic compressibility and compressive-tensile stresses along the cut direction are taken into account.

Key words: *characteristic size, boundary integral equation, linear elasticity, ideal elastoplastic model.*

Introduction. In studying the normally separating crack in the form of a mathematical cut within the framework of the Leonov–Panasyuk–Dugdale model, the plastic flow mechanism is postulated, with tensile stresses acting on the boundaries of the plastic zone equal to the yield stress [1, 2]. For different types of the plane state, the use of this postulate yield practically identical lengths of the plastic zone. Experimental data, however, show that these lengths are substantially different. Therefore, a correction is applied in the case of the plane strain state (formally, the yield stress increases by a factor of $\sqrt{3}$) [3], and the length of the plasticity zone becomes one third of its value in the plane stress state.

We consider a semi-discrete model of elastoplastic deformation of a plane with a semi-infinite straight-line cut of width δ_0 [4]. This scale is the minimum admissible level at which the hypotheses of mechanics of continuous media are valid [5, 6]. We assume that the plasticity zone is a rectangle of height δ_0 and length l_p, which has to be determined. The present work is aimed at determining the dependence of the length l_p on the type of the stress–strain state and on the magnitude of the external load. The results obtained are compared with the solution predicted by the Leonov–Panasyuk–Dugdale model.

1. Formulation and Solution of the Problem of Elastoplastic Deformation of the Layer in the Plane Strain State. We consider loading of a plane attenuated by a physical cut of width δ_0 by a symmetric external load (Fig. 1). The material lying on the continuation of the physical cut in the plane is assumed to form a layer of interaction with a homogeneous distribution of the stress–strain state over the layer thickness [5, 6]. Homogeneity of the stress–strain state is a consequence of the fact that the stress tensor components in the layer are averaged over the thickness:

$$\sigma_{ij}(x_2) = \frac{1}{\delta_0} \int\limits_{-\delta_0/2}^{\delta_0/2} \sigma_{ij}(x_1, x_2)\, dx_1 \qquad (i = 1, 2, 3, \quad j = 1, 2, 3).$$

Thus, the stress state of the layer is determined by the tensor of "averaged" stresses. The tangential stresses in this tensor are assumed to be negligibly small, as compared with diagonal components, for the type of loading

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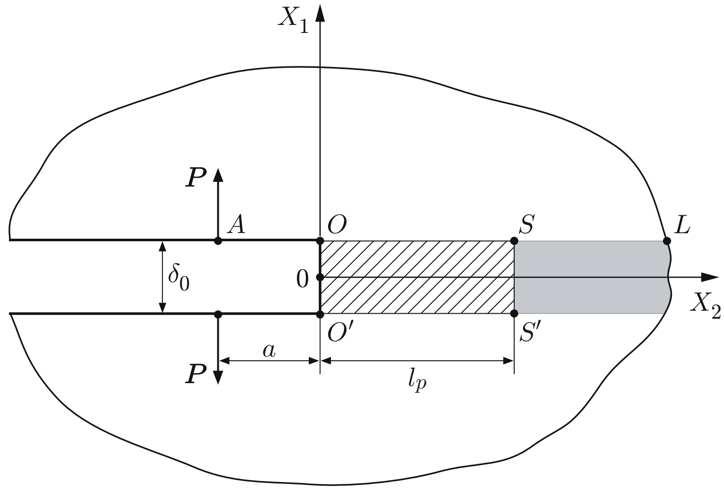

Fig. 1. Loading scheme.

considered here. We assume that the mean stress $\sigma_{22}(x_2)$ is caused by the tangential load on the layer boundary $\sigma_{21}(\pm\delta_0/2, x_2)$. In what follows, the stresses $\sigma_{11}(x_2)$, $\sigma_{22}(x_2)$, and $\sigma_{33}(x_2)$ are considered as the mean stresses in the layer, which are the principal stresses. In addition, the stress $\sigma_{21}(x_2) = \sigma_{21}(\delta_0/2, x_2)$ is also taken into account. The relation between the stresses and strains outside the layer is assumed to be described within the framework of the linear elasticity theory.

By virtue of problem symmetry, we consider only the upper half-plane ($x_1 \geqslant \delta_0/2$), and the action of the layer on this half-plane is replaced by the load

$$\boldsymbol{q}(x) = -(\hat{\sigma}_{11}\boldsymbol{e}_1 + \hat{\sigma}_{21}\boldsymbol{e}_2).$$

Here, $x \equiv x_2/\delta_0$ is the dimensionless coordinate, $\hat{\sigma}_{ij} = \beta\sigma_{ij}$ $(i = 1, 2,\ j = 1, 2)$ are the dimensionless stresses, $\beta = 2(1-\nu^2)/(\pi E)$ is the parameter of the material in the case of plane strain, E is Young's modulus, and ν is Poisson's coefficient.

The Flamant relations [7] connect the external loads $\hat{\sigma}_{11}$ and $\hat{\sigma}_{12}$ with the displacements of the half-plane boundary:

$$\hat{u}_1(x) = -\hat{P}\ln\Big(\frac{x+a}{l+a}\Big) + \int_0^l \hat{\sigma}_{11}(\xi)\,\ln\frac{|x-\xi|}{l-\xi}\,d\xi; \tag{1}$$

$$\hat{u}_2(x) = \int_0^l \hat{\sigma}_{12}(\xi)\,\ln\frac{|x-\xi|}{l-\xi}\,d\xi. \tag{2}$$

Here, $\hat{u}_i = u_i/\delta_0$ $(i = 1, 2)$ are the dimensionless displacements, $\hat{P} = P\beta/\delta_0$ is the dimensionless force per unit thickness, and l is the distance between the origin and the remote point L with zero displacement.

As the stress–strain state is homogeneous over the layer thickness, the equilibrium condition yields

$$\frac{\partial\hat{\sigma}_{22}}{\partial x} = -2\hat{\sigma}_{21}. \tag{3}$$

The displacements of the layer boundaries are determined from the conditions

$$\hat{u}_1(x) = \varepsilon_{11}(x)/2; \tag{4}$$

$$\hat{u}_2(x) = \int_l^x \varepsilon_{22}(x)\,dx. \tag{5}$$

In the plane strain state below the yield stress, the stresses and strains are related by Hooke's law

$$\varepsilon_{11} = A\hat{\sigma}_{11} - B\hat{\sigma}_{22}; \tag{6}$$

$$\varepsilon_{22} = A\hat{\sigma}_{22} - B\hat{\sigma}_{11}, \tag{7}$$

where $A = \pi/2$ and $B = \nu\pi/[2(1-\nu)]$ are dimensionless constants.

Let us differential Eq. (2) with respect to x:

$$\varepsilon_{22} = \frac{d\hat{u}_2}{dx} = \int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi. \tag{8}$$

Taking into account Eq. (4), we write Eq. (1) in the form

$$\varepsilon_{11} = -2\hat{P}\ln\Big(\frac{x+a}{l+a}\Big) + 2\int_0^l \hat{\sigma}_{11}(\xi)\ln\frac{|x-\xi|}{l-\xi}\,d\xi. \tag{9}$$

Using Eqs. (8) and (9), we find the change in the volume along the layer due to motion of the "walls" bounding its elastic space:

$$\int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi + 2\int_0^l \hat{\sigma}_{11}(\xi)\,\ln\frac{|x-\xi|}{l-\xi}\,d\xi - 2\hat{P}\ln\Big(\frac{x+a}{l+a}\Big) + \varepsilon_{33}(x) = \varepsilon_{11}(x) + \varepsilon_{22}(x) + \varepsilon_{33}(x) = \theta(x). \tag{10}$$

Note that Eq. (10) is universal and remains valid both for the elastic and elastoplastic behavior of the layer, because the change in the volume is assumed to be elastic.

Using Hooke's law (6), (7), and the condition $\varepsilon_{33} = 0$, we present equality (10) in the form

$$\int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi + 2\int_0^l \hat{\sigma}_{11}(\xi)\,\ln\frac{|x-\xi|}{l-\xi}\,d\xi - 2\hat{P}\ln\Big(\frac{x+a}{l+a}\Big) = (A-B)(\hat{\sigma}_{11} + \hat{\sigma}_{22}), \tag{11}$$

where the type of the plane state is determined by the constants A and B.

Equation (11) is supplemented by the condition of equality of the strains ε_{22} along the layer, which are calculated from Eq. (8) and directly from Hooke's law (7). As a result, the system of equations for the elastic zone acquires the form

$$\begin{gathered}\int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi + 2\int_0^l \hat{\sigma}_{11}(\xi)\,\ln\frac{|x-\xi|}{l-\xi}\,d\xi - 2\hat{P}\ln\Big(\frac{x+a}{l+a}\Big) = (A-B)(\hat{\sigma}_{11} + \hat{\sigma}_{22}),\\ A\hat{\sigma}_{22} - B\hat{\sigma}_{11} = \int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi, \qquad \frac{\partial\hat{\sigma}_{22}}{\partial x} = -2\hat{\sigma}_{12}.\end{gathered} \tag{12}$$

The main unknowns of system (12) are the stress tensor components. As the end-face plane of the initial cut is assumed to be unloaded, we have

$$\hat{\sigma}_{22}\Big|_{x=0} = 0. \tag{13}$$

Following [8], we assume in solving this problem that destruction of the solid is a discrete process; therefore, the stress state is assumed to be uniform within the interaction layer element of length δ_0 or of unit dimensionless length.

To construct the solution within the framework of the discrete model, we divide the half-plane boundary OL into n unit elements. Each element of the boundary k with the coordinates ξ_{k-1}, ξ_k, where $k = \overline{1, n}$, is characterized by constant (averaged over the element) stresses σ_{11}^k, σ_{22}^k, and σ_{12}^k, determined as follows:

$$\sigma_{ij}^k(x_k) = \int_{\xi_{k-1}}^{\xi_k} \hat{\sigma}_{ij}(\xi)\,d\xi.$$

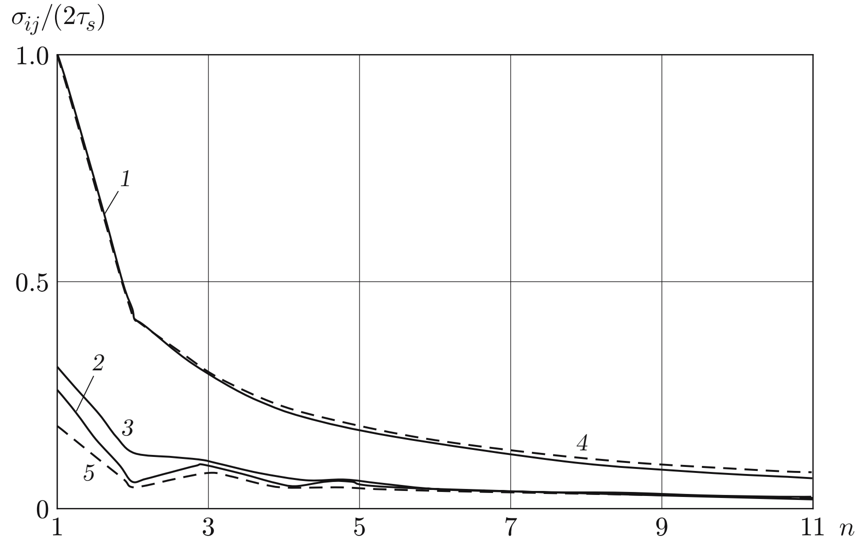

Fig. 2. Stresses in the layer during elastic deformation: curves 1–3 show σ_{11} (1), σ_{22} (2), and σ_{33} (3) in the plane strain state; curves 4 and 5 show σ_{11} (4) and σ_{22} (5) in the plane stress state ($\sigma_{33} \equiv 0$).

Here $x_k = (\xi_k + \xi_{k-1})/2$. The integrals in the equations of system (12) can be presented as the corresponding sums. For discretization of the equilibrium equation (3), it is integrated with respect to the kth element. As a result, we obtain $\sigma_{22}^k - \sigma_{22}^{k-1} = -2\sigma_{21}^k$. It should be noted that this approach is similar to the boundary element method [9] with permanent approximation, but differs from the latter method by the fact that it makes no sense to divide the boundary into smaller elements. As was noted above, the element size chosen here was determined by the fact that the problem is solved under the assumption of a continuous medium. Thus, the system of integral and differential equations (12) supplemented by the boundary condition (13) acquires the following form in the discrete presentation:

$$\sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi + 2\sum_{i=1}^{n} \sigma_{11}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \ln \frac{|x_i - \xi|}{l - \xi}\, d\xi - 2P \ln\Big(\frac{x_k + a}{n + a}\Big) = (A - B)(\sigma_{11}^k(x_k) + \sigma_{22}^k(x_k)),$$

$$A\sigma_{22}^k(x_k) - B\sigma_{11}^k(x_k) = \sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi, \qquad (14)$$

$$\sigma_{22}^k - \sigma_{22}^{k-1} = -2\sigma_{21}^k, \quad k = 1, \ldots, n, \qquad \sigma_{22}^0 = 0.$$

Note that the linear system (14) contains an infinite number of equations in the general case ($n \to \infty$). It is possible, however, to use a finite number of equations in analyzing the results. The numerical solution of the system displays a fairly food convergence: the results calculated at $n = 1000$ and 5000 differ by less than 1%. All further calculations were performed with $n = 5000$.

Curves 1, 2, and 3 in Fig. 2 show the stresses σ_{11}, σ_{22}, and σ_{33}, respectively, for the case of the plane strain state with the following problem parameters: $P = 1$, $a = 5$, and $\nu = 0.25$. Note that $\sigma_{22} = \sigma_{33} = 0$ for $\nu = 0$ and $\sigma_{22} < \sigma_{33}$ for the remaining admissible values of Poisson's coefficient in the case of the plane strain state.

When a certain criterion is reached, the material of the layer passes to the plastic state, which is described within the framework of an ideal elastoplastic model [10]. The criterion of the transition from the elastic to the plastic state is reaching the critical value of the maximum shear stress:

$$|\sigma_{ii} - \sigma_{jj}| = 2\tau_s. \qquad (15)$$

Here τ_s is the yield stress; $i = 1, 2, 3$, $j = 1, 2, 3$.

Under the plane strain conditions in the pre-fracture zone, the solution of the elastic problem yields $\sigma_{11} > \sigma_{33} \geqslant \sigma_{22}$ (see Fig. 2). Thus, the Tresca yield condition (15) for this type of loading is determined by the expression

$$\hat\sigma_{11} - \hat\sigma_{22} = 2\hat\tau_s, \tag{16}$$

where $\hat\tau_s = \beta\tau_s$ is the dimensionless yield stress.

We assume that the strains are low and that the following expansion is valid at the stage of elastoplastic deformation:

$$\varepsilon_{ii} = \varepsilon_{ii}^e + \varepsilon_{ii}^p, \qquad i = 1, 2 \tag{17}$$

(ε_{ii}^e and ε_{ii}^p are the elastic and plastic components of the total strain, respectively). The material is assumed to be plastically incompressible:

$$\varepsilon_{11}^p + \varepsilon_{22}^p + \varepsilon_{33}^p = 0. \tag{18}$$

In the plane strain state, the elastic and plastic components of the transverse strain are assumed to be equal to zero:

$$\varepsilon_{33}^e = 0, \qquad \varepsilon_{33}^p = 0. \tag{19}$$

Using Hooke's law (6), (7) and Eqs. (17)–(19), we can present the change in the volume as

$$\theta(x) = (A - B)(\hat\sigma_{11} + \hat\sigma_{22}). \tag{20}$$

From Eqs. (10) and (18)–(20), the condition of equilibrium of the layer element (3), and the Tresca yield condition (16), we obtain the following closed system of equations, which describes the deformation of the plastic zone of the layer of length l_p in the plane strain state:

$$\begin{gathered}\int\limits_0^l \hat\sigma_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi + 2\int\limits_0^l \hat\sigma_{11}(\xi)\,\ln\frac{|x-\xi|}{l-\xi}\,d\xi - 2\hat P\ln\Big(\frac{x+a}{l+a}\Big) = (A-B)(\hat\sigma_{11}+\hat\sigma_{22}),\\ \frac{\partial\hat\sigma_{22}}{\partial x} = -2\hat\sigma_{21}, \qquad \hat\sigma_{11} - \hat\sigma_{22} = 2\hat\tau_s.\end{gathered} \tag{21}$$

Here $x \leqslant l_p$.

Note that the conditions of identical stress vectors and of identical normal displacements are used for conjugation of the elastic and plastic zones at the boundaries OS and $O'S'$ (see Fig. 1).

In the elastic zone, where $x > l_p$ and $\hat\sigma_{11} - \hat\sigma_{22} < 2\hat\tau_s$, the stress–strain state is described by system (12). The main unknowns of systems (21) and (12) [with allowance for the boundary condition (13)] are the stress tensor components and the length of the plastic zone.

Based on system (21), (12), (13), we construct a discrete model that describes the elastoplastic deformation of the layer in the plane strain state. This model consists of the following three subsystems of equations.

1) Equations that describe the material behavior in the plastic zone [discrete analog of system (21)] on the interval $0 \leqslant x \leqslant l_p$:

$$\begin{gathered}\sum_{i=1}^n \sigma_{12}^i(x_i)\int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i-\xi}\,d\xi + 2\sum_{i=1}^n \sigma_{11}^i(x_i)\int\limits_{\xi_{i-1}}^{\xi_i} \ln\frac{|x_i-\xi|}{n-\xi}\,d\xi - 2P_{l+1}\ln\Big(\frac{x+a}{n+a}\Big) = \frac{\pi(\hat\tau_s+\hat\sigma_{22}^k)(1-2\nu)}{1-\nu},\\ \sigma_{22}^k - \sigma_{22}^{k-1} = -2\sigma_{21}^k, \quad \sigma_{11}^k - \sigma_{22}^k = 2\hat\tau_s, \quad k = 1,\ldots,l, \qquad \sigma_{22}^0 = 0\end{gathered} \tag{22}$$

(l is the number of elements in the plastic state).

2) Equations that describe the transition of the $(l+1)$th element from the elastic to the plastic state [discrete analog of system (12) under the condition that the stress in this element reaches the yield stress: $\hat\sigma_{11} - \hat\sigma_{22} = 2\hat\tau_s$]:

$$\sigma_{11}^k - \sigma_{22}^k = 2\hat{\tau}_s,$$

$$\sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi + 2\sum_{i=1}^{n} \sigma_{11}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \ln \frac{|x_i - \xi|}{l - \xi}\, d\xi - 2P_{l+1} \ln\Big(\frac{x+a}{n+a}\Big) = \frac{\pi(\sigma_{11}^k(x_k) + \sigma_{22}^k(x_k))(1-2\nu)}{2(1-\nu)}, \tag{23}$$

$$A\sigma_{22}^k(x_k) - B\sigma_{11}^k(x_k) = \sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi,$$

$$\sigma_{22}^k - \sigma_{22}^{k-1} = -2\sigma_{21}^k, \qquad k = l+1.$$

3) Equations that describe the material behavior in the elastic zone [discrete analog of system (12)]:

$$\sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi + 2\sum_{i=1}^{n} \sigma_{11}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \ln \frac{|x_i - \xi|}{l - \xi}\, d\xi - 2P_{l+1} \ln\Big(\frac{x+a}{n+a}\Big)$$
$$= \frac{\pi(\sigma_{11}^k(x_k) + \sigma_{22}^k(x_k))(1-2\nu)}{2(1-\nu)}, \tag{24}$$

$$A\sigma_{22}^k(x_k) - B\sigma_{11}^k(x_k) = \sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi,$$

$$\sigma_{22}^k - \sigma_{22}^{k-1} = -2\sigma_{21}^k, \qquad k = l+2, \ldots, n.$$

The total system of equations of the discrete deformation model, consisting of subsystems (22)–(24), contains $3n+1$ linear equations. The unknowns are $3n$ generalized stresses and the critical force P_{l+1} responsible for this stress state. We intend to solve this problem step by step, determining the critical force and the stress–strain state of the layer corresponding to reaching the yield criterion in the $(l+1)$th element at each step.

Figure 3 shows the distributions of the stresses normalized to the yield stress over the elements when the yield stress is reached in the second element (load P_2). It is seen that the stresses in a plastically deformed element under the plane strain condition can exceed the yield stress (see also [11]). This fact is explained by a rather high value of the hydrostatic component of the stress state.

2. Elastoplastic Deformation of the Layer in the Plane Stress State. In the plane stress state, the stress components are normalized to the parameter $\beta = 2/(\pi E)$, and Hooke's law has the form

$$\varepsilon_{11} = A\hat{\sigma}_{11} - B\hat{\sigma}_{22}, \qquad \varepsilon_{22} = A\hat{\sigma}_{22} - B\hat{\sigma}_{11}, \qquad \varepsilon_{33} = -B(\hat{\sigma}_{11} + \hat{\sigma}_{22}), \tag{25}$$

where $A = \pi/2$ and $B = \pi\nu/2$ are dimensionless constants.

The system of resolving equations for the elastic zone has the form (12). For the zone passing from the elastic to the plastic state, with allowance for $\hat{\sigma}_{33} = 0$, criterion (15) is accepted in the form $\hat{\sigma}_{11} = 2\hat{\tau}_s$. The condition of ultimate plasticity $\hat{\sigma}_{11} = \hat{\sigma}_{22}$ is assumed to be satisfied in the zone of developed plastic deformation [10, 12, 13]. Then, the yield condition (15) acquires the form

$$\hat{\sigma}_{11} = \hat{\sigma}_{22} = 2\hat{\tau}_s, \tag{26}$$

where $\hat{\tau}_s = \beta\tau_s$ is the dimensionless yield stress.

In view of the equilibrium equation (2), Eq. (26) yields zero tangential stresses at the layer boundary in the plastic zone: $\hat{\sigma}_{12} = 0$; therefore, we obtain the following relations in the plastic flow in the layer:

$$\hat{\sigma}_{11} = 2\hat{\tau}_s, \qquad \hat{\sigma}_{22} = 2\hat{\tau}_s, \qquad \hat{\sigma}_{12} = 0. \tag{27}$$

In the elastic zone, where $x > l_p$ and $\hat{\sigma}_{11} < 2\hat{\tau}_s$, system (27) is supplemented by system (12) with constant values of A and B for the plane stress state. As in the case of the plane strain state, the main unknowns of systems (27) and (12) are the stress tensor components and the plastic zone length. Note, in contrast to the plane strain

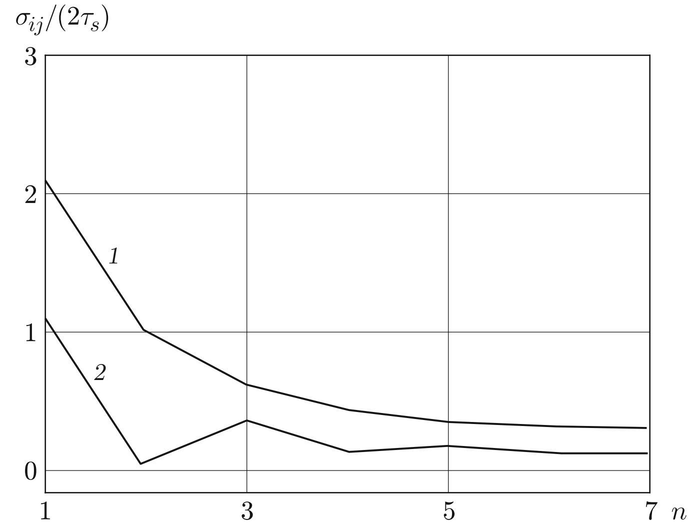

Fig. 3. Stresses in the layer during elastoplastic deformation in the plane strain state ($2\tau_s/E = 3 \cdot 10^{-3}$): curves 1 and 2 refer to σ_{11} and σ_{22}, respectively.

state, system (27) predicts that the stress state in homogeneous in the entire plastic zone and, hence, the strains are constant along the plastic zone and equal to the strains at the boundary of the transition of the layer from the elastic to the plastic state. The elastic strains of the layer are found from Eq. (25).

The discrete model constructed on the basis of Eqs. (27), (12) consists of the following three subsystems of equations.

1) Equations that describe the plastic zone [discrete analog of system (27)] are

$$\sigma_{11}^k = 2\hat{\tau}_s, \qquad \sigma_{22}^k = 2\hat{\tau}_s, \qquad \sigma_{21}^k = 0, \qquad k = 1, \ldots, l.$$

2) Equations that describe the transition of the $(l+1)$th element from the elastic to the plastic state [discrete analog of system (12) and condition of reaching the yield stress by the stress $\hat{\sigma}_{11}$ on the element] are

$$\sigma_{11}^k = 2\hat{\tau}_s,$$

$$\sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi + 2\sum_{i=1}^{n} \sigma_{11}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \ln \frac{|x_i - \xi|}{l - \xi}\, d\xi - 2P_{l+1} \ln\Big(\frac{x+a}{n+a}\Big) = \frac{\pi(\sigma_{11}^k(x_k) + \sigma_{22}^k(x_k))(1-\nu)}{2},$$

$$A\sigma_{22}^k(x_k) - B\sigma_{11}^k(x_k) = \sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi,$$

$$\sigma_{22}^k - \sigma_{22}^{k-1} = -2\sigma_{21}^k, \qquad k = l + 1.$$

3) Equations that describe the elastic zone [discrete analog of system (12)] are

$$\sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi + 2\sum_{i=1}^{n} \sigma_{11}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \ln \frac{|x_i - \xi|}{l - \xi}\, d\xi - 2P_{l+1} \ln\Big(\frac{x+a}{n+a}\Big) = \frac{\pi(\sigma_{11}^k(x_k) + \sigma_{22}^k(x_k))(1-\nu)}{2},$$

$$A\sigma_{22}^k(x_k) - B\sigma_{11}^k(x_k) = \sum_{i=1}^{n} \sigma_{12}^i(x_i) \int\limits_{\xi_{i-1}}^{\xi_i} \frac{1}{x_i - \xi}\, d\xi,$$

$$\sigma_{22}^k - \sigma_{22}^{k-1} = -2\sigma_{21}^k, \qquad k = l + 2, \ldots, n.$$

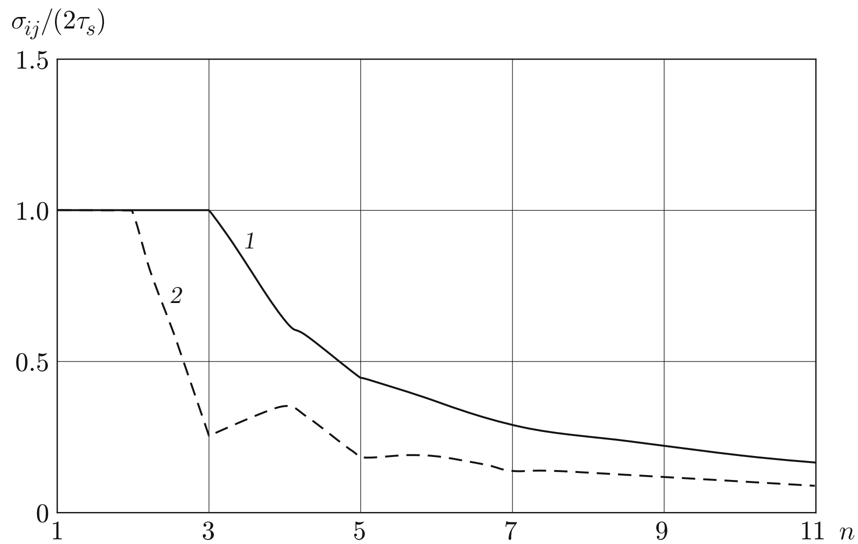

Fig. 4. Stresses in the layer during elastoplastic deformation in the plane stress state ($2\tau_s/E = 3 \cdot 10^{-3}$): curves 1 and 2 refer to σ_{11} and σ_{22}, respectively.

The distributions of the stresses on the element before reaching the plastic state is shown in Fig. 2. Curves 4 and 5 correspond to the stresses σ_{11} and σ_{22} at $P = 1$, $a = 5$, and $\nu = 0.25$. In the case of the elastic state, we obtain $\sigma_{11} > \sigma_{22} \geqslant \sigma_{33} = 0$ (see Fig. 2). Hence, the transition from the elastic to the plastic state is determined by the yield criterion (15) in the form $\hat{\sigma}_{11} = 2\hat{\tau}_s$.

Figure 4 shows the distribution of stresses normalized to the yield stress over the elements for the plane stress state with the third element passing to the plastic state (load P_3).

3. Comparison of the Approach Considered and the Leonov–Panasyuk–Dugdale Approach. The Leonov–Panasyuk–Dugdale model is based on the following assumptions [1, 2]:

1. The cut is considered as a mathematical cut; the plastic zone located on the continuation of the mathematical cut is a zone of zero thickness and of length l_p;

2. The stress state in the plastic zone is homogeneous tension with the stress equal to the tensile yield stress;

3. The elastic state outside the plastic zone is determined by asymptotic formulas of the linear elasticity theory. The presence of a plastic zone does not affect the law of stress distributions in the elastic zone;

4. The length of the plastic zone is calculated from the condition of finite stresses on the interface between the elastic and plastic zones.

Let us formalize the conditions, as applied to the problem considered:

(a) the thickness of the interaction layer is assumed to be zero: $\delta_0 = 0$;

(b) $\sigma_0 = \sigma_{11} = 2\tau_s$ at $0 \leqslant x_2 \leqslant l_p$ and $\sigma_{ij} = 0$ at $i \neq 1$ and $j \neq 1$.

To formalize conditions 3 and 4, we use the known asymptotic solutions given in [14, 15]. We denote the stress intensity factor under a wedge point load P applied to the semi-infinite cut by $K_{\rm I}^p$ and the corresponding stress intensity factor due to a constant-intensity external load σ_0 by $K_{\rm I}^0$. For condition 4 to be satisfied, we have to require that

$$\lim_{x_2 \to l_p} \sigma_{11}(0, x_2) = \text{const}. \qquad (28)$$

Condition (28) is valid is the stress intensity factors satisfy the requirement [16]

$$K_{\rm I}^p - K_{\rm I}^0 = 0. \qquad (29)$$

Using the results of [15], we find the expressions for the stress intensity factors in the problem considered:

$$K_{\rm I}^p = \sqrt{2}\, P/\sqrt{\pi(l_p + a)}, \qquad K_{\rm I}^0 = 2\sigma_0\sqrt{2l_p/\pi}. \qquad (30)$$

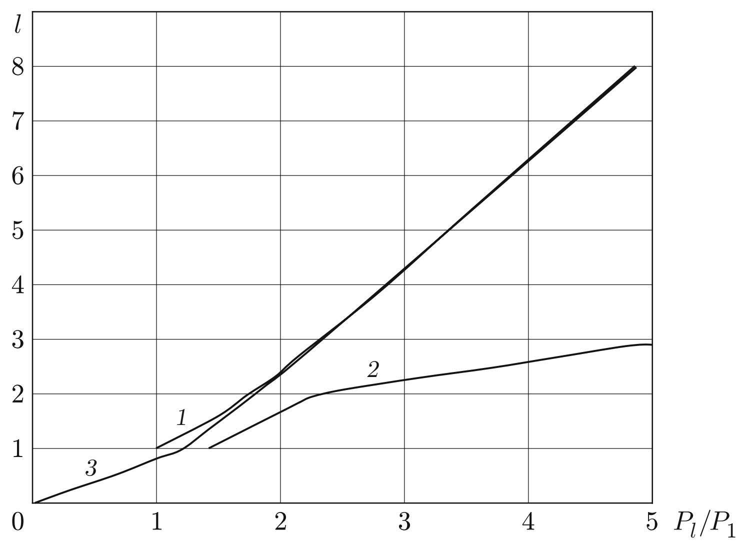

Fig. 5. Plastic zone length versus the external load: curves 1 and 2 are the results calculated with the use of the approach proposed here for the plane stress state (curve 1) and plane strain state (curve 2); curve 3 shows the results calculated by the Leonov–Panasyuk–Dugdale model in the plane stress state.

From conditions (29) and (30), we obtain the dependence between the wedge force and the plastic zone length:

$$P = 2\sigma_0\sqrt{l_p(l_p + a)}. \tag{31}$$

Figure 5 shows the length of the plastic zone as a function of the applied load, which was estimated by the approach proposed here and by the classical Leonov–Panasyuk–Dugdale model (P_1 is the critical load necessary for the first element to pass to the plastic state in the plane stress state and P_l is the critical load in the case of the plastic flow in the first l elements of the layer). It follows from Fig. 5 that the model proposed here allows the purely elastic behavior of the material to be described, in contrast to the Leonov–Panasyuk–Dugdale model. Curve 3 corresponds to dependence (31). It follows from the analysis of curves 1 and 2 that plastic deformation begins when the force P_1 reaches the critical value $P_1 = 1$. At the same time, it follows from the analysis of curve 3 that the Leonov–Panasyuk–Dugdale model predicts that the plastic zone appears under an arbitrarily small external load. Curves 1 and 2 are consistent with the experimental fact: the plastic zone length in the plane stress state is substantially greater than that in the plane strain state.

Conclusions. The model of discrete deformation allows the plastic zone evolution within a finite-thickness layer to be described both in the plane strain state and in the plane stress state.

The stress state of the layer and the plastic zone length are determined by solving appropriate boundary-value problems, which makes it possible (in contrast to the Leonov–Panasyuk–Dugdale approach) to take into account the redistribution of stresses in the elastic zone, induced by the increase in the plastic zone.

Allowance for compressive-tensile stresses and elastic compressibility in the plastic zone of the layer is found to be responsible for a significant difference in the laws of variation of the stresses and plastic zone lengths in the plane strain state and plane stress state.

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